that the lens is diffraction limited, i.e., the exit pupil has been chosen so that the image of a point source is the Airy intensity distribution.

But first, we remark that the intensity in this problem is related to that of the complementary case, the image of an opaque disc of radius a. According to *Babinet's principle*, the sum of the light amplitudes for these two cases is the constant light amplitude without either. (This is easy to see from the linearity of the Huyghens-Fresnel-Fraunhofer expression discussed in Appendix B 3). So, where one is light, the other is dark.

Suppose the hole is illuminated with incoherent light, as in ordinary microscopy. If $a < \lambda/4$, the illumination is nonetheless effectively coherent, since any incident plane wave of random phase will have little phase difference across the hole. If $a > \lambda/2$, the illumination may be considered incoherent. This is the case considered here.

1. Incoherent Illumination

If this were geometrical optics, light from each uniformly illuminated point of the object plane would pass through the lens and be focused as an illuminated point on the image plane. The properly scaled image of all these illuminated points would be a uniformly illuminated circle of radius a. We shall call the circumference of this circle the "image circle edge." The new wrinkle is that diffraction surrounds each imaged point with its own Airy disc (assuming that spherical aberration is negligible), so that the image extends beyond the image circle edge. The intensities add so, at a point $\mathbf{r}$ on the image plane, the net intensity is

$$I(r) \sim \frac{1}{\pi}\int_{A_0} dA_0 \left[\frac{J_1(k\tilde{b}|\mathbf{r}-\mathbf{r_0}|)}{|\mathbf{r}-\mathbf{r_0}|}\right]^2, \qquad \text{(G1)}$$

where A_0 is the area of the image circle, and $\tilde{b} \equiv b/f$ is called the numerical aperture.

For $a >> r_A$, where r_A is the Airy radius, the intensity at the center point of the image circle is, by (G1),

$$I(0) \sim \frac{1}{\pi}\int_0^a r_0 dr_0 \int_0^{2\pi} d\phi \left[\frac{J_1(k\tilde{b}r_0)}{r_0}\right]^2 \approx 1.$$

In this equation, the limit a has been extended to ∞ with no appreciable error, since the major contribution is from Airy discs centered within distance r_A of the origin.

As the point of interest moves off center, the intensity remains essentially constant, until at a distance $\approx a - r_A$ from the center, a distance r_A from the image circle edge. Then I starts to decrease, reaching the value $\approx .5$ at the edge. This is because, at the edge, ≈half the Airy discs contribute intensity, compared to the discs which contribute intensity at a point well inside the image circle.

Now, we turn to quantitative analysis of the general case, with no restriction of the relative sizes of a and r_A. We shall calculate the intensity (G1) outside the image circle, at $r = 0$ which is placed a distance z beyond the image circle edge, i.e., the center of the image circle in this coordinate system is at $r = a + z$. The contributing Airy disc centers lie within the image circle, between radius r_0 ($z \le r_0 \le 2a + z$) and radius $r_0 + dr_0$, along an arc subtending an angle 2ϕ. The hole circumference $(x - a - z)^2 + y^2 = a^2$ cuts this arc at two points. Setting $x = r_0 \cos\phi$ and $y = r_0 \sin\phi$ in this expression allows one to find $\cos\phi$. Eq. (G1) becomes

$$I_{\text{out}}(z) \sim \frac{2}{\pi}\int_z^{2a+z} dr_0 \cos^{-1}\left[\frac{r_0^2 + z^2 + 2az}{2r_0(a+z)}\right]\frac{J_1^2(k\tilde{b}r_0)}{r_0} \qquad \text{(G2)}$$

For completeness, we put here the comparable expression for the intensity inside the image circle. Again, we calculate the intensity (G1) at $r = 0$, where this new coordinate system origin is a distance z away from the center of the image circle. There are two contributions, one from a circular area of radius $a - z$, the other from the rest of the disc ($a - z \le r_0 \le a + z$):

$$\begin{aligned} I_{\text{in}}(z) \;&\sim\; 2\int_0^{a-z} dr_0 \frac{J_1^2(k\tilde{b}r_0)}{r_0} \\ &+ \frac{2}{\pi}\int_{a-z}^{a+z} dr_0 \cos^{-1}\left[\frac{r_0^2 + z^2 - a^2}{2r_0 z}\right]\frac{J_1^2(k\tilde{b}r_0)}{r_0}. \qquad \text{(G3)} \end{aligned}$$

For large a, (G2) becomes

$$I_{\text{out}}(z) \approx \frac{2}{\pi}\int_z^{\infty} dr_0 \cos^{-1}\left[\frac{z}{r_0}\right]\frac{J_1^2(k\tilde{b}r_0)}{r_0}$$

This is a function of $k\tilde{b}z = 3.83(z/r_A)$. Numerical evaluation shows $I_{\text{out}}(z)$ drops from .5 at $z = 0$ to $\approx .05$ at $z = r_A$. While it is somewhat subjective, this suggests that we take the perceived edge of the image of the hole to be located where the intensity is 5% of its maximum value at the center of the image circle. Thus, diffraction increases the radius of a large hole from a to $R \approx a + r_A$.

By changing the variable of integration in (G2) to r_0/a, one sees that the intensity is a function of two variables, z/a and $k\tilde{b}a/3.83 = a/r_A$. For each value of a/r_A, one can numerically solve Eq. (G2) for the value of z/a for which $I(z) = .05I(0)$. This is the ratio R/a, where R is defined as the radius of the image. A graph of R/r_A vs a/r_A is given in Fig. 12, and is discussed in section III H.

[1] P. W. van der Pas, Scientiarum Historia **13**, 127 (1971): see [68] for a more detailed discussion.

[2] A transcript of an interesting historical talk about

the Lewis and Clark expedition and its genesis, by Robert S Cox, given at Monticello in 2004, may be found at http://www.monticello.org/streaming /speakers/transcripts/cox.html

[3] *Original Journals of the Lewis and Clark Expedition, 1804-1806* (Dodd, Mead and Company, New York, 1905), edited by R. G. Thwaites, Volume 5, p.95.

[4] A wealth of material on the Lewis and Clark expedition and related topics appears at http://www.lewis-clark.org/. For a discussion of Clarkia pulchella, a picture of the plant, and Pursh's drawing of it which appeared in his *Flora*, click on "Natural History," then "Plants," then "Clarkia." It is also interesting to read here about the lives (and tragic deaths of the first three) of Lewis (see also http://www.prairieghosts.com/meriwet.html), Pursh (see also J. Ewan, Proc. Amer. Phil. Soc, **96** # 5, 599 (1952)), available at Googlebooks), Douglas (see also http://www.coffeetimes.com/daviddouglas.htm) and Barton, all by the American botanist J. Reveal.

[5] J. Loudon, *The Ladies Flower Garden of Ornamental Annuals* (W. Smith, London 1842), pp. 56-57 and her beautiful illustration of Clarkia plants between these two pages. It may be viewed at http://books.google.com/. A biography of Jane Loudon may be found at http://www.shigitatsu.com/LOUDON%201.htm.

[6] D. Douglas, *Journal kept by David Douglas during his travels in North America 1823-1827* (W. Wesley and son, London 1914), W. Wilks and H. R. Hutchinson eds. This is a marvelous first-hand adventure story. It can be read or downloaded at the Washington State historic site, http://www.secstate.wa.gov/history/publications.aspx (click on Exploration and Early Travel).

[7] Ibid, p. 57.

[8] Ibid, pp. 130-131.

[9] Ibid, p. 146.

[10] We are indebted to Bronwen Quarry of the Hudson's Bay Company archives for furnishing information about the dates of travel and fate of the William and Ann.

[11] Ibid, pp. 71-72.

[12] D. J. Mabberley, *Jupiter Botanicus: Robert Brown of the British Museum* (J. Cramer, Braunschweig 1985). The biographical details given in this paper are almost exclusively from this authoritative source.

[13] Wikipedia contains a short biography of Robert Brown, as does the Australian dictionary of Biography, http://www.adb.online.anu.edu.au/biogs/A010149b.htm, each with pictures of Brown, old and young respectively. See also http://www.answers.com/topic/robert-brown.

[14] B. J. Ford, http://www.brianjford.com/wbbrowna.htm. This web site also contains a photograph of the Bancks-made Linnean Society microscope of Brown's, as well as a comparison of a view with a single lens and a compound microscope of the time, and a short video of Brownian motion of milk fat droplets.

[15] http://www.microscopy-uk.org.uk/dww/home /hombrown.htm also has a video of milk fat droplets undergoing Brownian motion, of sizes .5 to $3\mu m$, with tips on how to duplicate the observation. It incorrectly states: "... he noticed the motion in tiny particles suspended within the medium of living pollen grains." although correctly adding "Some texbooks, even to university level, incorrectly state that he observed the movement of the pollen grains themselves. Most pollen grains are too large to exhibit noticeable Brownian motion."

[16] Robert M. Mazo, *Brownian Motion: Fluctuations, Dynamics and Applications* (Clarendon Press, Oxford 2002). The first few pages, available at googlebooks, http://books.google.com/, and also at Amazon, http://www.amazon.com/, contains a brief treatment of the first two sections of this paper. See also Edward Nelson, *Dynamical Theories of Brownian Motion* (Princeton University Press, Princeton 1967), which is available at http://www.math.princeton.edu/ nelson/books/bmotion.pdf, Chapter 2. Both are excellent texts on "mathematical" Brownian motion.

[17] A biography of John Lindley is available at the orchid information web site http://www.orchids.co.in/: select orchidolologists.

[18] The Horticultural Society garden in Chiswick was part of the 655 acre estate of the lessee, William Spencer Cavendish (1790-1858), the 6th Duke of Devonshire, with a gate to the garden installed for the Duke's use. The physicist Henry Cavendish (1731-1810) was related, a grandson of the 2nd Duke. What remains of this estate, Chiswick House on 68 acres, a fine example of Palladian architecture, is being restored and can be visited (http://www.chgt.org.uk). More of the acreage exists as a Chiswick public park, Dukes Meadows (http://www.dukesmeadowstrust.org/). However, the Horticultural Society garden is no more, replaced by city streets.

[19] We are indebted to David Mabberley for "strongly" suggesting that the slips catalog be looked into, and to Armando Mendez, of the Botany Library of the Natural History Museum, for finding this sheet and mailing us a copy.

[20] We are indebted to Brent Elliot, Historian of the Royal Horticultural Society, for this information. The society does not have any surviving plant receipt books for the garden at Chiswick until the 1840s, so one cannot track Clarkia pulchella into or out of the garden. Lindley was in overall supervision of the gardens, with Donald Munro as head gardener, and a staff of under-gardeners. It was common for botanists to share plants of interest.

[21] The complete entries for both dates are reproduced in J. Ramsbottom, Journ. of Bot. **70** 13, (1932).

[22] E. Small, I. J. Bassett, C. W. Crompton, and H. Lewis, *Pollen Phylogeny in Clarkia*, Taxon **20**, 739 (1971) contains detailed dimensional measurements on many species of Clarkia, including pulchella.

[23] R. Brown, Edinburgh New Philos. Journ. **5** 358, (1828) and Philos. Mag. **4** 161, (1828). This, and the addendum published a year later, Edinburgh Journ. Sci **1** 314, (1829), can be downloaded from the New York Botanical Gardens, at sciweb.nybg.org/science2/pdfs/dws/Brownian.pdf.

[24] B. J. Ford, *Single Lens: the Story of the Simple Microscope* (Harper and Row, New York 1985), pp. 152-153.

[25] B. J. Ford, Microscopy **34** 406, (1982); The Linnean **1** 12, (1985). These describe the loving restoration of the Linnean Society microscope, with the first paper giving quantitative measurements on the lenses. See also B. J. Ford, Infocus **15**, 18 (2009).

[36] Mabberley, Ibid, p. 389.

[27] This is a translation from the French. The original appears in Mabberley, Ibid, pp. 270-271.

[28] H. Loncke, "Making a von Leeuwenhoek microscope

lens," Microscope **138** (April, 2007): this is an online article available at http://www.microscopy-uk.org.uk. Select the Library/Issue Archive.

[29] J. J. Lister, Phil. Trans. Roy. Soc. London, **120** 187, (1830).

[30] W. A. Newman in *History of Carcinology* (CRC Press, 1993), F. Truesdale, ed., p. 357, available at Googlebooks.

[31] D. Layton, Journ. Chem. Ed. **42**, 367 (1965).

[32] A casual on–line search turns up many examples. J. Bernstein Am. Jour. Phys. **74**, 863 (2006), in a fine, authoritative assessment of Einstein's 1905 Brownian motion paper and its impact, nonetheless writes on p.85, about Brown: "He noticed that if pollen grains of a few ten-thousands of a centimeter were suspended in water, they executed an incessant jigging motion." The size is right, but it is not the Clarkia pollen size; Einstein year website, http://www.einsteinyear.org/facts: "In 1827 the biologist Robert Brown noticed that if you looked at pollen grains in water through a microscope, the pollen jiggles about."; Announcement of Gauss Prize (2006) for Kiyoshi Itô, http://www.mathunion.org/Prizes/Gauss/index.html.
"In Itô's case, this way begins with a look into a microscope showing pollen grains or dust particles in water moving around in an erratic way."; T. Spencer, power point lecture "Random Walk from Einstein to the Present" at the educational arm of the Institute for Advanced Study at Princeton, http://pcmi.ias.edu/, type in Brownian motion: "Around 1827 Brown made a systematic study of the swarming motion of microscopic particles of pollen."; Wikipedia Dictionary, http://en.wiktionary.org/wiki/Brownian, "1. Pertaining to botanist Robert Brown (1773-1858), who investigated the movement of pollen suspended in water."; http://www.wisegeek.com/what-is-brownian-motion.htm, "As a botanist, Brown first observed the effect in pollen floating in water, where it is visible with the naked eye." ; Univ. of St. Andrews, School of Mathematics and Statistics, http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Ito.html, "Brown, a botanist, discovered the motion of pollen particles in water."; http://physicsworld.com/cws/article/print/21146, "Most of us probably remember hearing about Brownian motion in high school, when we are taught that pollen grains jiggle around randomly in water due the impacts of millions of invisible molecules."; http://www.math.utah.edu/ carlson/teaching/java/prob/brownianmotion/1/, "In 1827 the English botanist Robert Brown noticed that pollen grains suspended in water jiggled about under the lens of the microscope, following a zigzag path like the one pictured below."

[33] Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford 1983), sixth (corrected) edition, p. 234.

[34] B. J.Ford, The Microscope **40**, 235 (1992) verified that Brown could have seen motion with his Linnean x170 lens, writing: "The Clarkia pollen was obtained from anthers of C. pulchella at the Botanical Garden at Cambridge University, and pollen specimens from other species within the Oenotheraceae were also utilized. Exactly as Brown recorded, the experiments were carried out in the month of June and the pollen grains were mounted in water after removal from pre-dehiscent anthers. ... The phenomenon of Brownian movement was well resolved by the original microscope lens. Within the pollen grains, ceaseless movement could be observed." However, while the author saw the contents of the pollen, he did not see them undergoing Brownian motion, as this appears to suggest. Rather, with this lens, he observed the Brownian motion of similarly sized milk fat droplets (private communication from B. J. Ford). Presumably, what was meant is that "ceaseless movement *could have been* observed" with that lens. Perhaps this led to the erroneous statement in an article reporting on this work, Science News **142** (Aug. 15), 109 (1992): "Each grain contains a thick liquid ... When Brown looked inside the pollen grains with his microscope, he could see tiny particles, each about 1 micron across, suspended in the liquid and constantly in motion."

[35] At the web site of the Scuola Normale of Pisa, http://gbamici.sns.it/eng/osservazioni, select osservazionibiologiche.htm.

[36] Mabberley, Ibid.

[37] John Lindley, *An Introduction to Botany*, Fourth edition, Volume 1 (Longman, Brown Green and Longmans, London 1848), p. 361, available at the Googlebooks site.

[38] J. Heslop-Harrison, *Pollen: Development and Physiology* (Butterworths, London 1971).

[39] A. Frey-Wyssling, *Die Pflanzliche Zellwand* (Springer-Verlag, Berlin 1959).

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[46] J. P. Mascarenhas, Am. J. Bot. **53**, 563 (1966).

[47] W. G. Rosen, S. R. Gawlik, W. W. Dashek and K. A. Siegesmond, Am. J. Bot. **51**, 61 (1964).

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[51] D. E. Bilderback, Environ. Health Perspect. **37**, 95 (1981).

[52] T. Schumucker, Planta **23**, 264 (1935).

[53] D. B. Dickinson, J. Am. Soc. Hort. Sci. **103**, 263 (1978).

[54] J. L. Brewbaker and B. H. Kwack, Am. J. Bot. **50**, 859 (1963).

[55] J. P. Mascarenhas, Ibid.

[56] D. E. Bilderback, Ibid.

[57] M. H. Weisenseel and L. F. Jaffe, Planta **133**, 1 (1976).

[58] K. R. Robinson and M. A. Messerli, Sci. STKE **2002**, pe51 (2002). The article in this electronic journal may be downloaded from the web site of one of the authors: http://www.biology.purdue.edu /people/faculty/robinson/Lab/publications.htm.

[59] X. Chao-Tian, Q. Yi-Lan , C. Su-Hong and T. Hui-Qiao, J. Pl. Physiol. and Molec. Biol. **31**, 53 (2005).

[60] N. A. Eckardt, The Plant Cell bf 17, 327 (2005).

[61] Here are some web sites that describe how to observe pollen tubes in the laboratory: http://www-saps.plantsci.cam.ac.uk/pollen/pollen2.htm,

http://www-saps.plantsci.cam.ac.uk/worksheets/ssheets/ssheet4.htm, and http://www.microscopy-uk.org.uk/mag/artdec99/jgpollen.html.

[62] H. Lewis and M. E. Lewis, *The Genus Clarkia* (University of California Publications in Botany **20**, pp. 241-392, 1955), p. 356.

[63] I grew Clarkia plants from Diane's seeds, http://www.dianeseeds.com: packets of C. pulchella (containing ≈1000 seeds), C. amoena (≈1800 seeds) and C. elegans (≈1500 seeds) each cost $2.00. Everwilde farms, http://www.everwilde.com sells packets (≈2000 seeds) of the latter two species for $2.50. Thompson and Morgan, http://www. tmseeds.com, a British company, sells packets of C. pulchella and C. elegans (≈400 seeds) for $2.55. Monticello sells a packet of seeds of C. pulchella (a plant cultivated by Thomas Jefferson) for $2.50, but I did not have good results with these seeds.

[64] The U. S. Department of Agriculture site, http://plants.usda.gov/, has much general information on species. Under Scientific Name, type in Clarkia, and then choose Clarkia Pursh.

[65] http://www.leevalley.com. The Lee Valley Seed Starter costs $22.50 plus shipping. One can purchase 25 plastic pots for $9.50, and a plastic tray which holds 24 such pots for $26.50.

[66] Hydrofarm Green Thumb or Jump Start (they seem to be the same) fixtures with bulbs are available from various vendors. For example, DirtWorks, http://www.dirtworks.net/Grow-Lights.html, sells the fixture, the 2 foot version (which will light one Lee Valley Seed Starter) costs $69 plus shipping and the 4-foot version (which I bought and which lights three Lee Valley Seed Starters) costs $89 plus shipping.

[67] ImageJ is available from the National Institutes of Health web site http://rsbweb.nih.gov/ij/.

[68] P. W. van der Pas, Scientiarum Historia **13**, 127 (1971): Of Brown's molecules, van der Pas says: " ... they were approximately of the same size; their diameter varying between 1.26 and 1.6 microns. These statements are not true, BROWN was led to them because he worked with an imperfect lens at the border line of its magnifying power." This seems to be the only place to find this latter assertion which, however, van der Pas did not enlarge upon. He wrote this paper to call attention to a rather throw-away paragraph in a paper in 1784 by Jan Ingenhousz, thereby intimating Ingenhousz's priority over Brown. The purpose of Ingenhousz's paper was to introduce the idea of a transparent cover slip in microscopy to prevent water evaporation. Unlike others who had seen Brownian motion before Brown, but attributed it to life, Ingenhousz observed and clearly asserted in this paragraph that nonliving matter underwent the motion, but he did no systematic investigation.
Citing van der Pas, Mabberley (Op. Cit. p. 272) says: "It has been shown that Brown's 'molecules' were artifacts, there being no particles 1.26-1.6 μm across in pollen grains or elsewhere." However, this statement is only partially correct. While there are not *universal* particles of this size range as Brown supposed, spherosomes imaged to such size were certainly seen by Brown. Spherosomes have been observed in various plant tissues of diameter .4-4 μm[69].

[69] T. J. Jacks, L. Y. Yatsu and A. M. Altschul, Plant Physiol. **42**, 585 (1967), L. Y. Yatsu, T. J. Jacks and T. P Hensarling, Plant Physiol. **48**, 675 (1971).

[70] K. W. A. Strehl, Zeit. fur Instr. **22**, 213 (1902). The Strehl criterion is mentioned in most books on optical design, e.g., D. Malacara and Z. Malacara, *The Handbook of Optical Design, 2nd Edition* (Marcel Dekker, New York 2003), p. 211.

[71] F. Perrin, Journ. Phys. radium **V**, 497 (1934); R. Vasanthi, S. Rarichandran and B. Bagchi, Journ. Chem. Phys. **114**, 7989 (2001); Y. Han et. al., Science **314**, 626 (2006).

[72] The polystyrene latex spheres were obtained from Ted Pella, Inc., http://www.tedpella.com/.

[73] Two web sites that describe construction of such microscopes are http://www.microscopy-uk.org.uk/mag/artjul06/aa-lens3.html and http://www.funsci.com/fun3_en/usph/usph.htm.

[74] Edmund Optical Company, 1 mm diameter ball lens #NT43-708, costing $22.

[75] A. C. Hardy and F. H. Perrin, *The Principles of Optics* (McGraw-Hill, New York 1932), p. 58, Eq. (71).

[76] A. Einstein, Ann. d. Phys. **17**, 549 (1905).

[77] P. M. Morse *Thermal Physics* (Benjamin, New York 1969), p. 228; C. Kittel *Elementary Statistical Physics* (Wiley, New York 1958), p. 153; a recent paper in this journal, D. Jia et. al., Am. J. Phys **75**, 111 (2007) (see also textbook references therein).

[78] K. Huang *Statistical Mechanics* (Wiley, New York 1965), p. 119 and and L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford 1978), p.63: the latter also contains a problem that discusses the viscous torque on a sphere, p. 68.

[79] The viscous force on an ellipsoid moving parallel to an axis is discussed in Lamb's classic volume, H. Lamb *Hydrodynamics* (Dover, New York 1945), section 339. The viscous torque on an ellipse rotating about an axis is calculated in a 1922 paper, G. B. Jeffery, Proc. Roy. Soc. London, series A **102**, 161(1922), Eq. (36).

[80] See chapter IX of M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford 1983), sixth (corrected) edition, p. 459.

[81] J. D. Jackson *Classical Electrodynamics* (Wiley, New York 1975).

[82] M. Born and E. Wolf, Ibid, p. 387.

[83] A simpler problem is the scattering of a plane wave by a sphere. The solution for our problem (scattering of a wave emerging from a point source) is a superposition of solutions of the plane wave problem. However, this "simpler" problem is itself quite complicated. Its solution can be written exactly, as an infinite sum of angular momentum eigenstates, each with a spherical Bessel function giving the radial behavior. Since $\lambda << R$, one cannot truncate the series at a few terms. Sophisticated techniques (such as the Watson transform, Regge pole theory, method of steepest descent) are used to sum appropriate terms corresponding to the physical behavior of rays, first discussed by Debye. The first sum corresponds to reflection from the sphere, the second to the geometrical optics refraction and its attendant aberrations, the third to one internal reflection (responsible for the behavior of the rainbow), the fourth to two internal reflections (responsible for the behavior of the glory), etc: H. M. Nussenzveig, Journ. Math. Phys. **10**, 82 & 125 (1969). The sum for the electromagnetic field is called the Mie solution. Mie solution calculators, which sum the terms

numerically, are available on the web. For an analytic treatment, see W. T. Grandy Jr., *Scattering of waves from large spheres* (Cambridge U. P., Cambridge 2005).

[84] H. M. van Horn and E. E. Salpeter, Phys. Rev. **157**, 751(1967): this discusses the pioneering 1928 work of J. H. van Vleck.

[85] Born and Wolf, Ibid, opposite p. 478. See also http://en.wikipedia.org/wiki/Spherical_aberration.